DIOPHANTINE EQUATION $X^4 + Y^4 = 2(U^4 + V^4)$

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Abstract. In this paper, the theory of elliptic curves is used for finding the solutions of the quartic Diophantine equation $X^4 + Y^4 = 2(U^4 + V^4)$.

1. Itroduction

The solubility in integers of diophantine equation

$$a_0X^4 + a_1Y^4 + a_2U^4 + a_3V^4 = 0 \tag{1.1}$$

has been considered by many mathematicians, where a_0, a_1, a_2 and a_3 are nonzero integers. The most famous and simplest one is proposed by Euler (see [6] page 201) for the constants $a_0 = a_1 = 1$ and $a_2 = a_3 = -1$. Euler gave a two-parameter solutions for this equation. Zajta [13] applied several methods including the Pythagorean and algebraic reduction method, for parametrization of $A^4 + B^4 = C^4 + D^4$. Brudno [1] and Lander [8] gave new parametrizations for like wise power diophantine equations, specially for $A^4 + B^4 = C^4 + D^4$. Using geometric methods and the property of tangent plane, Richmond [9] parameterized the equation (1.1), for the case that the product $a_0a_1a_2a_3$ is a square number. Setting $a_0 = 1, a_1 = 4, a_2 = -1$ and $a_3 = -4$, Choudhry [2] presented two-parameter solutions of the equation . Choudhry [3], has considered a special family of Diophantine equation by

$$A^4 + hB^4 = C^4 + hD^4, \tag{1.2}$$

and has found a list of integer solutions for cases $h \leq 101$. Noam Elkies [5] found infinitely many solutions of equation (1.1) by taking $a_0 = a_1 = a_2 = 1$ and $a_3 = -1$. In his method he used the theory of elliptic curves . This paper is concerned with the integral solutions of (1.1) where $a_0 = a_1 = 1$ and $a_2 = a_3 = -2$, i.e.,

$$X^4 + Y^4 = 2(U^4 + V^4). \tag{1.3}$$

The smallest known solution for this equation is $(X, Y, U, V) = (21, 19, 20, 7)$. When we say the smallest solution we mean the smallest up to sign. For example $(21, 19, 20, -7)$ is a solution but it is not a new one. We give infinitely many

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solutions of (1.3) by means of a specific congruent number elliptic curve namely $y^2 = x^3 - 36x^2$.
First, let us recall some basic facts about elliptic curves. An elliptic curve E over $\mathbb{Q}$ is a curve that is given by an equation of the form $y^2 = x^3 + ax + b$, where $a, b \in \mathbb{Q}$. By the Mordell-Weil theorem, the rational points on an elliptic curve form a finitely generated abelian group, which is denoted by $E(\mathbb{Q})$ and so one can write the following decomposition

$$E(\mathbb{Q}) \simeq E(\mathbb{Q})_{\mathrm{tors}} \oplus \mathbb{Z}^r, \tag{1.4}$$

where r is a nonnegative integer called the *rank* of E and $E(\mathbb{Q})_{\mathrm{tors}}$ is the finite group consisting of all the elements of finite order in $E(\mathbb{Q})$ [11].
A positive square free integer n is called a congruent number if it is the area of some right triangle with rational sides. The following theorem tells us that whether a number is congruent or not.

Theorem 1.1. *Consider the elliptic curve $E_n : y^2 = x^3 - n^2x$. n is a congruent number if and only if the elliptic curve $E_n(\mathbb{Q})$ has a positive rank.*

Proof. See [7] □

In (1.3), let us change X to $U + t$ and Y to $U - t$ where t is a parameter. Therefore we have the equation $6t^2U^2 + t^4 = V^4$. Now taking $Z = tU$ yields

$$6Z^2 = V^4 - t^4. \tag{1.5}$$

Having said that, the following proposition is useful for our purpose.

Proposition 1.1. *Let c be a nonzero integer. The equation $X^4 - Y^4 = cZ^2$ has a solution with $XYZ \neq 0$ if and only if $|c|$ is a congruent number. More precisely, if $X^4 - Y^4 = cZ^2$ with $XYZ \neq 0$ then $E_c : y^2 = x^3 - c^2x$ with $(x, y) = (-cY^2/X^2, c^2YZ/X^3)$, and conversely if $E_c : y^2 = x^3 - c^2x$ with $y \neq 0$ then $X^4 - Y^4 = cZ^2$, with*

$$X = x^2 + cx - c^2, \quad Y = x^2 - 2cx - c^2, \quad \textit{and} \quad Z = 4y(x^2 + c^2).$$

Proof. See section 6.5 proposition 6.5.6 of [4]. □

According to the equation (1.3), theorem 1.1 and proposition 1.1, we see that the equation (1.5) has a solution, since $c = 6$ is a congruent number. So, equation (1.3) has a solution. To find this solution we use the transformations of proposition 1.1. From (x, y) on elliptic curve $E_6 = x^3 - 36x$, we obtain

$$t = x^2 - 12x - 36,$$

$$V = x^2 + 12x - 36,$$

$$U = \tfrac{4y(x^2+36)}{t}.$$

Therefore, $(U+t, U-t, U, V)$ is a rational solution of (1.3). Multiplying this solution by t, we eliminate denominator of U. Next, we let $x=b/e^2$ and $y=c/e^3$ for some integers b, c, e. Substituting these x and y and multiplying all equations by e^8 we get the following integer solutions for (1.3).

$$\begin{aligned}
X &= b^4+1296e^8+864be^6+72b^2e^4+144ce^5-24b^3e^2+4b^2ce,\\
Y &= -864be^6-b^4-1296e^8-72b^2e^4+144ce^5+24b^3e^2+4b^2ce,\\
U &= 4(b^2+36e^4)ce,\\
V &= (b^2-36e^4-12be^2)(b^2-36e^4+12be^2).
\end{aligned} \tag{1.6}$$

Remark 1. *Note that, the additive inverse of a point (x,y) on E_6 is $(x,-y)$. This means that we change c to $-c$ in (1.6). Consequently, If (X,Y,U,V) is a solution obtained from (x,y), then $(-X,-Y,-U,V)$ is a solution obtained from $(x,-y)$, which is not a new one up to sign.*

2. Numerical Results

In this section we obtain primitive solutions of the diophantine equation (1.3).

Definition 2.1. A solution (A,B,C,D) of the diophantine equation (1.1) is said to be primitive if $\gcd(A,B,C,D)=1$.

Using SAGE software [10], we see that $Rank(E_6(\mathbb{Q}))=1$ and $P=(-3,9)$ is the generator of non-torsion subgroup of $E_6(\mathbb{Q})$. So, without taking into consideration of the inverse points, we know that every point of the form $(X,Y)=n(-3,9)$ for some $n\in\mathbb{N}$, is also a non-torsion point in $E(\mathbb{Q})$. we have $n(-3,9)=\left(\frac{\phi_n(-3,9)}{{\psi^2}_n(-3,9)}, \frac{\omega_n(-3,9)}{{\psi^3}_n(-3,9)}\right)$ where ψ_n is the n-th division polynomial of E_6, $\phi_n = x{\psi^2}_n-\psi_{n+1}\psi_{n-1}$ and $\omega_n=(4y)^{-1}(\psi_{n+2}{\psi^2}_{n-1}-\psi_{n-2}{\psi^2}_{n+1})$ (For more details see [12] pages 81-84). Therefor, we can set $e=\psi_n(-3,9)$, $b=\phi_n(-3,9)$ and $c=\omega_n(-3,9)$ in Eq (1.6) to obtain a sequence of solution (X_n,Y_n,U_n,V_n). For simplicity we omit $(-3,9)$ to obtain

$$\begin{aligned}
X_n &= {\phi_n}^4+1296{\psi_n}^8+864\phi_n{\psi_n}^6+72{\phi_n}^2{\psi_n}^4+144\omega_n{\psi_n}^5\\
&\quad -24{\phi_n}^3{\psi_n}^2+4{\phi_n}^2\omega_n\psi_n,\\
Y_n &= -864\phi_n{\psi_n}^6-{\phi_n}^4-1296{\psi_n}^8-72{\phi_n}^2{\psi_n}^4+144\omega_n{\psi_n}^5\\
&\quad +24{\phi_n}^3{\psi_n}^2+4{\phi_n}^2\omega_n\psi_n,\\
U_n &= 4({\phi_n}^2+36{\psi_n}^4)\omega_n\psi_n,\\
V_n &= ({\phi_n}^2-36{\psi_n}^4-12\phi_n{\psi_n}^2)({\phi_n}^2-36{\psi_n}^4+12\phi_n{\psi_n}^2).
\end{aligned} \tag{2.1}$$

Not all the solutions of (X_n, Y_n, U_n, V_n) are primitive and some of them are multiples of $(21, 19, 20, 7)$. For instance $(-3, 9)$ leads to $(189, 171, 180, -63) = 9(21, 19, 20, -7)$, which is not a new one.
Let $(A_n, B_n, C_n, D_n) = (X_n/d_n, Y_n/d_n, U_n/d_n, V_n/d_n)$, in which $d_n = \gcd(X_n, Y_n, U_n, V_n)$. Obviously, $\{(A_n, B_n, C_n, D_n)\}$ is a sequence of primitive solutions of diophantine equation (1.3). Using SAGE, we computed (A_n, B_n, C_n, D_n) for $2 \leq n \leq 1000$ and presented some of new primitive solutions in the following.

$$
\begin{aligned}
&n = 2,\\
&A_2 = 988521,\\
&B_2 = -1661081,\\
&C_2 = -336280,\\
&D_2 = -1437599.
\end{aligned}
$$

$$
\begin{aligned}
&n = 3,\\
&A_3 = -22394369951939,\\
&B_3 = -59719152671941,\\
&C_3 = -41056761311940,\\
&D_3 = 43690772126393.
\end{aligned}
$$

$$
\begin{aligned}
&n = 4,\\
&A_4 = 500901052196260108859464 1,\\
&B_4 = -959074737626305392403761,\\
&C_4 = 202496789216814784809544 0,\\
&D_4 = 415611880854896794176960 1.
\end{aligned}
$$

$$
\begin{aligned}
&n = 5,\\
&A_5 = 3851034625881084687405424604570750401 01,\\
&B_5 = -58316597151277440454625613485820959901,\\
&C_5 = 16339343271841551414295842348562704010 0,\\
&D_5 = -31849720982909420672712416881546090080 7.
\end{aligned}
$$

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